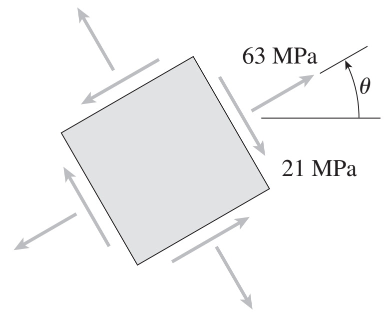
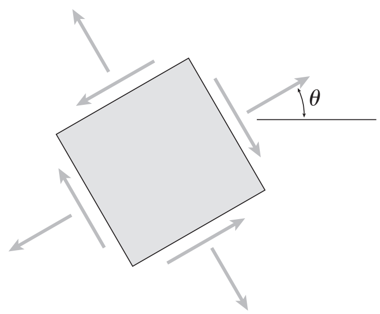
Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 63$ MPa and a shear stress $\tau_\theta = -21$ MPa on a certain inclined plane (see figure).

Determine the stresses acting on all faces of a stress element oriented at $\theta = 30°$ and show the stresses on a sketch of the element.



Solution 2.6-16 **Bar in uniaxial stress**



$\sigma_\theta = 63$ MPa $\qquad \tau_\theta = -21$ MPa

INCLINED PLANE AT ANGLE θ

$$\sigma_\theta = \sigma_x \cos^2\theta$$

$$63 \text{ MPa} = \sigma_x \cos^2\theta$$

$$\sigma_x = \frac{63 \text{ MPa}}{\cos^2\theta} \qquad (1)$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$

$$-21 \text{ MPa} = -\sigma_x \sin\theta\cos\theta$$

$$\sigma_x = \frac{21 \text{ MPa}}{\sin\theta\cos\theta} \qquad (2)$$

Equate (1) and (2):

$$\frac{63 \text{ MPa}}{\cos^2\theta} = \frac{21 \text{ MPa}}{\sin\theta\cos\theta}$$

or

$$\tan\theta = \frac{21}{63} = \frac{1}{3} \qquad \theta = 18.43°$$

From (1) or (2): σ_x=70.0 MPa (tension)

STRESS ELEMENT AT $\theta = 30°$

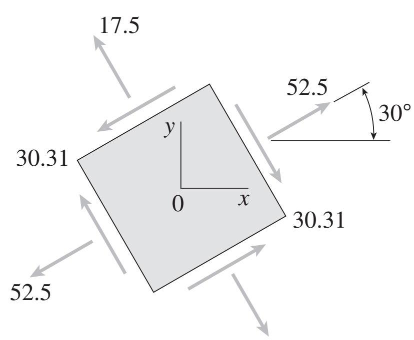
$$\sigma_\theta = \sigma_x \cos^2\theta = (70 \text{ MPa})(\cos 30°)^2$$
$$= 52.5 \text{ MPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= (-70 \text{ MPa})(\sin 30°)(\cos 30°)$$
$$= -30.31 \text{ MPa}$$

Plane at $\theta = 30° + 90° = 120°$

$$\sigma_\theta = (70 \text{ MPa})(\cos 120°)^2 = 17.5 \text{ MPa}$$

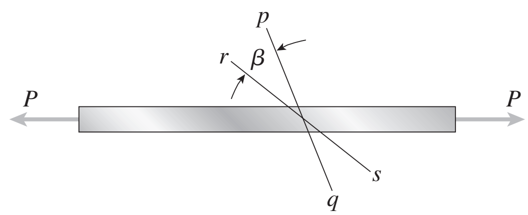
$$\tau_\theta = (-70 \text{ MPa})(\sin 120°)(\cos 120°)$$
$$= 30.31 \text{ MPa}$$



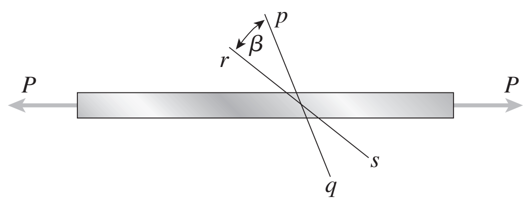
NOTE: All stresses have units of MPa.

Problem 2.6-17 The normal stress on plane *pq* of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane *rs*, which makes an angle $\beta = 30°$ with plane *pq*, the stress is found to be 2500 psi.

Determine the maximum normal stress σ_{max} and maximum shear stress τ_{max} in the bar.



Solution 2.6-17 **Bar in tension**



Eq. (2.29a):

$\sigma_\theta = \sigma_x \cos^2\theta$

$\beta = 30°$

PLANE *pq*: $\sigma_1 = \sigma_x \cos^2\theta_1$ $\qquad \sigma_1 = 7500$ psi

PLANE *rs*: $\sigma_2 = \sigma_x \cos^2(\theta_1 + \beta)$ $\qquad \sigma_2 = 2500$ psi

Equate σ_x from σ_1 and σ_2:

$$\sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \qquad \text{(Eq. 1)}$$

or

$$\frac{\cos^2\theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \qquad \frac{\cos\theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (2):

$$\frac{\cos\theta_1}{\cos(\theta_1 + 30°)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$\theta_1 = 30°$

MAXIMUM NORMAL STRESS (FROM EQ. 1)

$$\sigma_{max} = \sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{7500 \text{ psi}}{\cos^2 30°}$$

$$= 10{,}000 \text{ psi} \longleftarrow$$

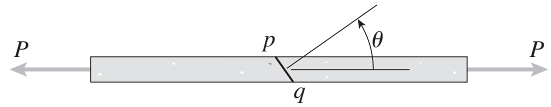
MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{\sigma_x}{2} = 5{,}000 \text{ psi} \longleftarrow$$

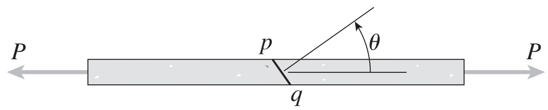
Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane *pq* (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45°. The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.

(a) Determine the angle θ so that the bar will carry the largest load *P*. (Assume that the strength of the glued joint controls the design.)

(b) Determine the maximum allowable load P_{max} if the cross-sectional area of the bar is 225 mm².



Solution 2.6-18 **Bar in tension with glued joint**



$25° < \theta < 45°$

$A = 225\ \text{mm}^2$

On glued joint: $\sigma_{allow} = 5.0\ \text{MPa}$

$\tau_{allow} = 3.0\ \text{MPa}$

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2\theta \qquad \sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{5.0\ \text{MPa}}{\cos^2\theta} \qquad (1)$$

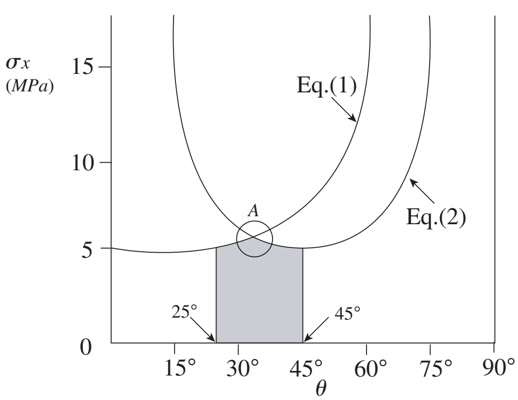
$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$

Since the direction of τ_θ is immaterial, we can write:
$|\tau_\theta| = \sigma_x \sin\theta\cos\theta$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin\theta\cos\theta} = \frac{3.0\ \text{MPa}}{\sin\theta\cos\theta} \qquad (2)$$

GRAPH OF EQS. (1) AND (2)



(a) DETERMINE ANGLE θ FOR LARGEST LOAD

Point *A* gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point *A*, we equate Eqs. (1) and (2).

$$\frac{5.0\ \text{MPa}}{\cos^2\theta} = \frac{3.0\ \text{MPa}}{\sin\theta\cos\theta}$$

$$\tan\theta = \frac{3.0}{5.0} \qquad \theta = 30.96° \longleftarrow$$

(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

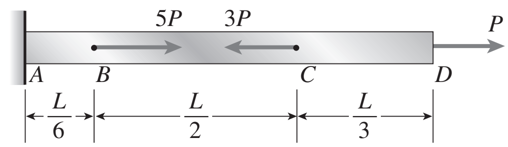
$$\sigma_x = \frac{5.0\ \text{MPa}}{\cos^2\theta} = \frac{3.0\ \text{MPa}}{\sin\theta\cos\theta} = 6.80\ \text{MPa}$$

$$P_{max} = \sigma_x A = (6.80\ \text{MPa})(225\ \text{mm}^2)$$
$$= 1.53\ \text{kN} \longleftarrow$$

Strain Energy

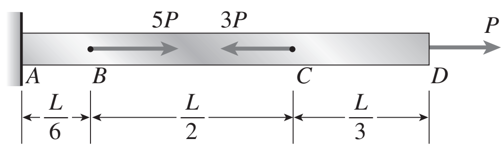
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar *AD* of length *L*, cross-sectional area *A*, and modulus of elasticity *E* is subjected to loads 5*P*, 3*P*, and *P* acting at points *B*, *C*, and *D*, respectively (see figure). Segments *AB*, *BC*, and *CD* have lengths *L*/6, *L*/2, and *L*/3, respectively.



(a) Obtain a formula for the strain energy *U* of the bar.
(b) Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.^{2}, and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 Bar with three loads



$P = 6$ k

$L = 52$ in.

$E = 10.4 \times 10^6$ psi

$A = 2.76$ in.^{2}

INTERNAL AXIAL FORCES

$N_{AB} = 3P \qquad N_{BC} = -2P \qquad N_{CD} = P$

LENGTHS

$L_{AB} = \frac{L}{6} \qquad L_{BC} = \frac{L}{2} \qquad L_{CD} = \frac{L}{3}$

(a) STRAIN ENERGY OF THE BAR (EQ. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{1}{2EA}\left[(3P)^2\left(\frac{L}{6}\right) + (-2P)^2\left(\frac{L}{2}\right) + (P)^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{P^2L}{2EA}\left(\frac{23}{6}\right) = \frac{23P^2L}{12EA} \quad \longleftarrow$$

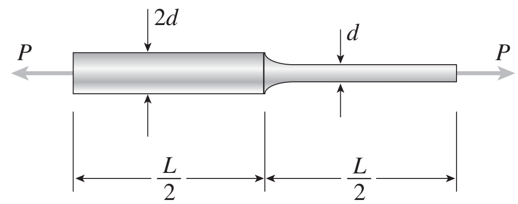
(b) SUBSTITUTE NUMERICAL VALUES:

$$U = \frac{23(6\text{ k})^2(52\text{ in.})}{12(10.4 \times 10^6\text{ psi})(2.76\text{ in.}^2)}$$

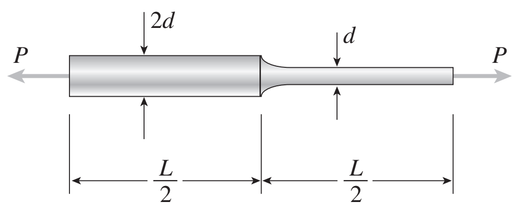
$$= 125\text{ in.-lb} \quad \longleftarrow$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E.

(a) Obtain a formula for the strain energy U of the bar due to the load P.
(b) Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 **Bar with two segments**



(a) STRAIN ENERGY OF THE BAR

Add the strain energies of the two segments of the bar (see Eq. 2-40).

$$U = \sum_{i=1}^{N} \frac{N_i^2 L_i}{2 E_i A_i} = \frac{P^2(L/2)}{2E}\left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)}\right]$$

$$= \frac{P^2 L}{\pi E}\left(\frac{1}{4d^2} + \frac{1}{d^2}\right) = \frac{5P^2 L}{4\pi E d^2} \quad \longleftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$P = 27$ kN $\qquad L = 600$ mm

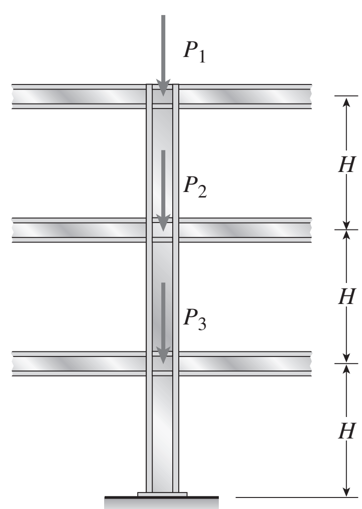
$d = 40$ mm $\qquad E = 105$ GPa

$$U = \frac{5(27 \text{ kN})^2(600 \text{ mm})}{4\pi(105 \text{ GPa})(40 \text{ mm})^2}$$

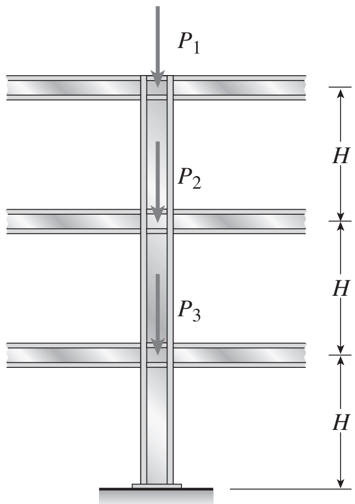
$$= 1.036 \text{ N} \cdot \text{m} = 1.036 \text{ J} \quad \longleftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^{2}, and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 **Three-story column**



$H = 10.5$ ft $\quad E = 30 \times 10^6$ psi

$A = 15.5$ in.^{2} $\quad P_1 = 40$ k

$P_2 = P_3 = 60$ k

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-40).

Upper segment: $N_1 = -P_1$

Middle segment: $N_2 = -(P_1 + P_2)$

Lower segment: $N_3 = -(P_1 + P_2 + P_3)$

STRAIN ENERGY

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{H}{2EA}\left[P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2\right]$$

$$\downarrow$$

$$= \frac{H}{2EA}[Q]$$

$[Q] = (40\text{ k})^2 + (100\text{ k})^2 + (160\text{ k})^2 = 37{,}200\text{ k}^2$

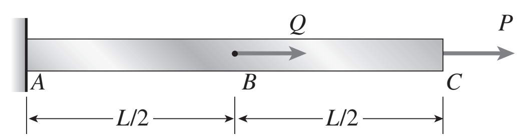
$2EA = 2(30 \times 10^6\text{ psi})(15.5\text{ in.}^2) = 930 \times 10^6\text{ lb}$

$$U = \frac{(10.5\text{ ft})(12\text{ in./ft})}{930 \times 10^6\text{ lb}}[37{,}200\text{ k}^2]$$

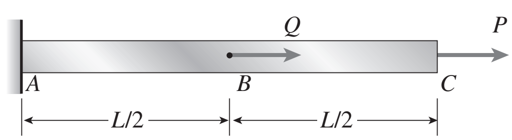
$$= 5040\text{ in.-lb} \quad \longleftarrow$$

Problem 2.7-4 The bar *ABC* shown in the figure is loaded by a force *P* acting at end *C* and by a force *Q* acting at the midpoint *B*. The bar has constant axial rigidity *EA*.

(a) Determine the strain energy U_1 of the bar when the force *P* acts alone ($Q = 0$).
(b) Determine the strain energy U_2 when the force *Q* acts alone ($P = 0$).
(c) Determine the strain energy U_3 when the forces *P* and *Q* act simultaneously upon the bar.



Solution 2.7-4 **Bar with two loads**



(a) FORCE *P* ACTS ALONE ($Q = 0$)

$$U_1 = \frac{P^2 L}{2EA} \quad \longleftarrow$$

(b) FORCE *Q* ACTS ALONE ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(c) FORCES *P* AND *Q* ACT SIMULTANEOUSLY

Segment *BC*: $U_{BC} = \dfrac{P^2(L/2)}{2EA} = \dfrac{P^2 L}{4EA}$

Segment *AB*: $U_{AB} = \dfrac{(P + Q)^2(L/2)}{2EA}$

$$= \frac{P^2 L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2 L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if *Q* is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density ($lb/in.^3$)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 **Strain-energy density**

DATA:

Material	Weight density ($lb/in.^3$)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

STRAIN ENERGY PER UNIT VOLUME

$$U = \frac{P^2L}{2EA} \qquad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$\mu = \frac{U}{V} = \frac{\sigma^2}{2E}$$

At the proportional limit:

$\mu = \mu_R$ = modulus of resistance

$$\mu_R = \frac{\sigma_{PL}^2}{2E} \qquad \text{(Eq. 1)}$$

STRAIN ENERGY PER UNIT WEIGHT

$$U = \frac{P^2L}{2EA} \qquad \text{Weight } W = \gamma AL$$

γ = weight density

$$\mu_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

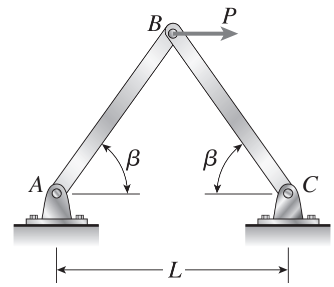
$$\mu_W = \frac{\sigma_{PL}^2}{2\gamma E} \qquad \text{(Eq. 2)}$$

RESULTS

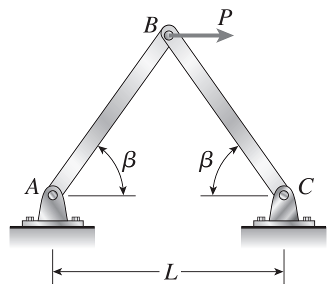
	μ_R (psi)	μ_W (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss *ABC* shown in the figure is subjected to a horizontal load *P* at joint *B*. The two bars are identical with cross-sectional area *A* and modulus of elasticity *E*.

(a) Determine the strain energy *U* of the truss if the angle $\beta = 60°$.
(b) Determine the horizontal displacement δ_B of joint *B* by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 Truss subjected to a load *P*



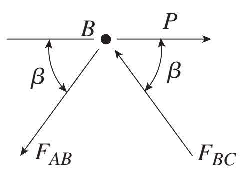
$\beta = 60°$

$L_{AB} = L_{BC} = L$

$\sin\beta = \sqrt{3}/2$

$\cos\beta = 1/2$

FREE-BODY DIAGRAM OF JOINT *B*



$\Sigma F_{vert} = 0 \quad \uparrow_+ \quad \downarrow^-$

$-F_{AB}\sin\beta + F_{BC}\sin\beta = 0$

$F_{AB} = F_{BC}$ (Eq. 1)

$\Sigma F_{horiz} = 0 \quad \underset{+}{\rightarrow} \quad \underset{-}{\leftarrow}$

$-F_{AB}\cos\beta - F_{BC}\cos\beta + P = 0$

$$F_{AB} = F_{BC} = \frac{P}{2\cos\beta} = \frac{P}{2(1/2)} = P \qquad \text{(Eq. 2)}$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i} = \frac{(N_{AB})^2 L}{2EA} + \frac{(N_{BC})^2 L}{2EA}$$

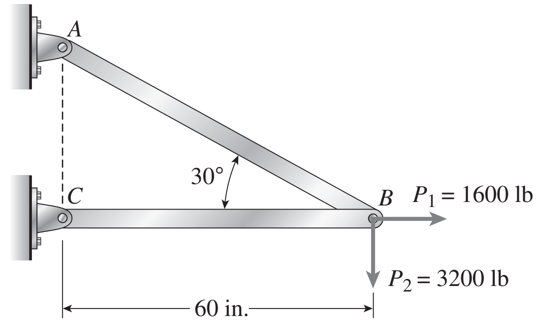
$$= \frac{P^2 L}{EA} \quad \longleftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT *B* (EQ. 2-42)

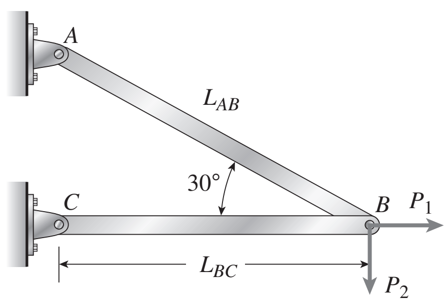
$$\delta_B = \frac{2U}{P} = \frac{2}{P}\left(\frac{P^2 L}{EA}\right) = \frac{2PL}{EA} \quad \longleftarrow$$

Problem 2.7-7 The truss *ABC* shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.^{2} and are made of steel with $E = 30 \times 10^6$ psi.

(a) Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
(b) Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
(c) Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 Truss with two loads



$P_1 = 300$ lb

$P_2 = 900$ lb

$A = 2.4$ in.^{2}

$E = 30 \times 10^6$ psi

$L_{BC} = 60$ in.

$\beta = 30°$

$$\sin \beta = \sin 30° = \frac{1}{2}$$

$$\cos \beta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30°} = \frac{120}{\sqrt{3}} \text{ in.} = 69.282 \text{ in.}$$

$$2EA = 2(30 \times 10^6 \text{ psi})(2.4 \text{ in.}^2) = 144 \times 10^6 \text{ lb}$$

FORCES F_{AB} AND F_{BC} IN THE BARS

From equilibrium of joint *B*:

$$F_{AB} = 2P_2 = 1800 \text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300 \text{ lb} - 1558.8 \text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	−1558.8 lb	−1258.8 lb

(a) LOAD P_1 ACTS ALONE

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300 \text{ lb})^2(60 \text{ in.})}{144 \times 10^6 \text{ lb}}$$

$$= 0.0375 \text{ in.-lb} \longleftarrow$$

(b) LOAD P_2 ACTS ALONE

$$U_2 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\left[(1800 \text{ lb})^2(69.282 \text{ in.}) + (-1558.8 \text{ lb})^2(60 \text{ in.})\right]$$

$$= \frac{370.265 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} = 2.57 \text{ in.-lb} \longleftarrow$$

(c) LOADS P_1 AND P_2 ACT SIMULTANEOUSLY

$$U_3 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\left[(1800 \text{ lb})^2(69.282 \text{ in.}) + (-1258.8 \text{ lb})^2(60 \text{ in.})\right]$$

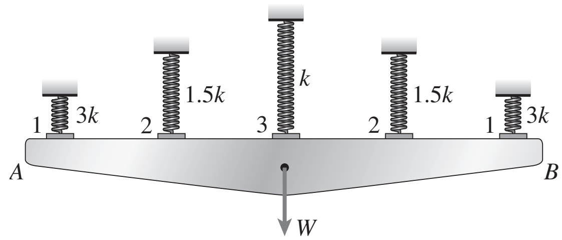
$$= \frac{319.548 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}}$$

$$= 2.22 \text{ in.-lb} \longleftarrow$$

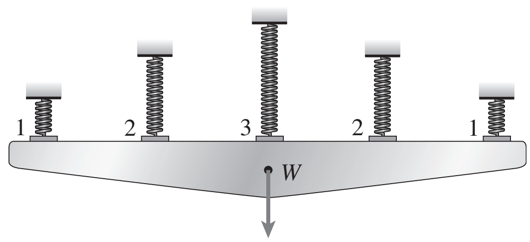
NOTE: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar *AB* supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k, respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar *AB*, which has weight W, causes the springs to elongate by an amount δ.

(a) Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
(b) Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W.
(c) Determine the forces F_1, F_2, and F_3 in the springs.
(d) Evaluate the strain energy U, the displacement δ, and the forces in the springs if $W = 600$ N and $k = 7.5$ N/mm.



Solution 2.7-8 Rigid bar supported by springs



$k_1 = 3k$

$k_2 = 1.5k$

$k_3 = k$

δ = downward displacement of rigid bar

For a spring: $U = \dfrac{k\delta^2}{2}$ Eq. (2-38b)

(a) Strain energy U of all springs

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2}$$

$$= 5k\delta^2 \longleftarrow$$

(b) Displacement δ

Work done by the weight W equals $\dfrac{W\delta}{2}$

$\downarrow$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \longleftarrow$$

(c) Forces in the springs

$$F_1 = 3k\delta = \frac{3W}{10} \qquad F_2 = 1.5k\delta = \frac{3W}{20} \longleftarrow$$

$$F_3 = k\delta = \frac{W}{10} \longleftarrow$$

(d) Numerical values

$W = 600\,\text{N} \quad k = 7.5\,\text{N/mm} = 7500\,\text{N/mm}$

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4\ \text{N} \cdot \text{mm} = 2.4\ \text{J}$$

$$\delta = \frac{W}{10k} = 8.0\ \text{mm} \longleftarrow$$

$$F_1 = \frac{3W}{10} = 180\ \text{N} \longleftarrow$$

$$F_2 = \frac{3W}{20} = 90\ \text{N} \longleftarrow$$

$$F_3 = \frac{W}{10} = 60\ \text{N} \longleftarrow$$

Note: $W = 2F_1 + 2F_2 + F_3 = 600\,\text{N}$ (Check)